CONTACT PROBLEM ON THE INTERACTION BETWEEN AN ELASTIC DISK AND TWO DIFFERENT RIGID STAMPS

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A plane contact problem for an elastic disk interacting with two different rigid stamps is considered. The disk-stamp system is equilibrated by a force applied to the center of the disk. It is assumed that there are no friction forces in the domain of contact, and no loading outside of it.

The mentioned problem is reduced to determining the contact pressures from a system of integral equations of the first kind. An asymptotic solution of the system is constructed. Examples are considered.

This paper is a further development and extension of part of the results elucidated in [1].

1. Formulation of the problem. Reduction to a system of integral equations. Asymptotic solution of the system of integral equations. Let an elastic medium fill a circular domain S of radius R (Fig. 1), interact with two different rigid stamps exerting impressive forces Q_1 and Q_2. A force $Q = Q_2 - Q_1$ is applied at the center of the domain S. Let us assume there are no friction forces in the contact domain and no loading outside of it. In this case the following conditions can be written on the boundary of the domain S:

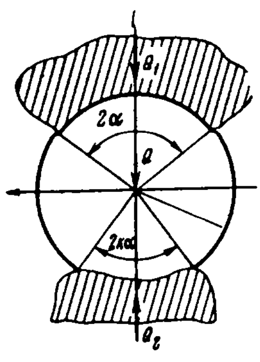

Fig. 1

$$\sigma_r = 0 \quad \text{for} \quad -\pi + k\alpha < \theta < -\alpha, \ \alpha < \theta < \pi - k\alpha$$

$$\tau = 0 \quad \text{for} \ |\theta| \leqslant \pi; \quad u_r = f_1(\theta) \quad \text{for} \ |\theta| \leqslant \alpha$$

$$u_r = f_2(\theta) \quad \text{for} \ \pi - k\alpha \leqslant \theta \leqslant \pi + k\alpha$$

We assume the following relative to the function $f_i(\theta)$:

$$f_1(\theta) = f_1(-\theta) \quad \text{for} \ |\theta| \leqslant \alpha,$$

$$f_2(\theta + \pi) = f_2(-\theta + \pi) \quad \text{for} \ |\theta| \leqslant k\alpha \tag{1.1}$$

Under the assumptions (1.1) made, evidently

$$\sigma_r(r, \theta) = \sigma_r(r, -\theta), \tau(r, \theta) = -\tau(r, -\theta).$$

Now, if known methods of solving the plane problem of elasticity theory are utilized [2,3], then the problem formulated can easily be reduced to determining the contact pressures from the following kind of system of integral equations:

$$\int_{-\alpha}^{\alpha} q(\varphi) K(\varphi - \theta) d\varphi + \int_{-k\alpha}^{k\alpha} q_1(\varphi + \pi) K(\varphi - \theta - \pi) d\varphi = \pi\Delta R^{-1} f_1(\theta) - \pi C \cos\theta, \ |\theta| \leqslant \alpha \tag{1.2}$$

$$\int_{-k\alpha}^{k\alpha} q_1(\varphi + \pi) K(\varphi - \theta) d\varphi + \int_{-\alpha}^{\alpha} q(\varphi) K(\varphi - \theta - \pi) d\varphi =$$

$$= \pi\Delta R^{-1} f_2(\theta + \pi) + \pi C \cos\theta, \quad |\theta| \leqslant k\alpha$$

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$$K(t) = -\frac{1}{2} - \frac{1}{16}(5 - 8\nu)(1 - \nu)^{-1}\cos t - \cos t \ln|2 \sin t/2| + \frac{1}{4}(1 - 2\nu)(1 - \nu)^{-1}$$
$$(\pi \operatorname{sgn} t - t)\sin t, \quad \Delta = \frac{1}{2} E (1 - \nu^2)^{-1}, \quad C = \mathrm{const}, \quad t = \varphi - \theta \tag{1.3}$$

Without restricting the generality, we henceforth consider $k \leqslant 1$.

Let us interchange variables in the system (1.2) so that the integrals on the right sides of the equations would have integration limits -1 to 1, and let us replace θ by αx in the first equation of (1.2) and θ by $k\alpha x$ in the second. Then let us represent the system (1.2) as an equivalent system of integral equations of the second kind, and obtain

$$\omega^{(i)}(x) = \frac{P_i}{\pi} - \frac{\Delta}{\pi R} \int_{-1}^{1} \frac{\sqrt{1 - t^2}}{t - x} g_i'(\delta_i \alpha t)\, dt + \frac{\alpha \delta_i}{\pi^2} \times$$
$$\times \int_{-1}^{1} \frac{\sqrt{1 - t^2}}{t - x}\, dt \int_{-1}^{1} (a_{20} \operatorname{sgn}(t - y) + F'[\alpha \delta_i (t - y)]) \frac{\omega^{(i)}(y)}{\sqrt{1 - y^2}}\, dy + \tag{1.4}$$
$$+ \frac{\alpha \delta_j}{\pi^2} \int_{-1}^{1} \frac{\sqrt{1 - t^2}}{t - x}\, dt \int_{-1}^{1} K'[\alpha \delta_i (t - k \delta_i^{-2} y)] \frac{\omega^{(i)}(x)}{\sqrt{1 - y^2}}\, dy, \quad |x| \leqslant 1 \quad (i \neq j,\ i,\ j = 1, 2)$$

where

$$P_i = \int_{-1}^{1} \frac{\omega^{(i)}(x)}{\sqrt{1 - x^2}}\, dx, \qquad q(\alpha x) = \frac{\omega^{(1)}(x)}{\sqrt{1 - x^2}}, \qquad q_1(k\alpha x + \pi) = \frac{\omega^{(2)}(x)}{\sqrt{1 - x^2}}$$
$$g_1'(t) = f_1'(\alpha t) + R\Delta^{-1} C \sin \alpha t, \quad g_2'(k\alpha t) = f_2'(k\alpha t + \pi) - R\Delta^{-1} C \sin \alpha t \tag{1.5}$$
$$F(\tau) = K(\tau) + \ln|\tau| - a_{20}|\tau|, \qquad \tau = t - y, \qquad a_{20} = \frac{1}{4}(1 - 2\nu)(1 - \nu)^{-1}$$
$$(\delta_1 = 1,\ \delta_2 = k)$$

Let us briefly describe the process of passing from (1.2) to (1.4). Taking account of the evident inequality $\alpha(1 + k) < \pi$ it is easy to note that the function $K(\theta - \varphi)$ has only a logarithmic singularity on the line $\theta = \varphi$ and the function $K(\theta - \varphi - \pi)$ is continuous in the domain of variation of its argument. Therefore, each equation of (1.2) can be represented as

$$\int_{-\beta}^{\beta} \varphi(t) \ln|t - x|\, dt = \int_{-\beta}^{\beta} \varphi(t) \psi(t - x)\, dt + \int_{-\gamma}^{\gamma} \varphi_1(t) \psi_1(t - x)\, dt + g(x), \qquad |x| \leqslant \beta$$
$$\psi(t - x) \in C(-2\gamma, 2\gamma), \qquad \psi_1(t - x) \in C(-\gamma - \beta, \gamma + \beta)$$

Following [4], the latter is easily written as an equivalent integral equation of the second kind.

If it is assumed that

$$f(\alpha x) \in H_p^{\beta}(-1, 1), \qquad p \geqslant 1, \qquad \beta > 0$$
$$f_2(k\alpha x + \pi) \in H_m^{\gamma}(-1, 1), \qquad m \geqslant 1, > \gamma > 0$$

(here and henceforth $H_n^{\alpha}(-\beta, \beta)$ denotes the space of functions whose n-th derivative satisfies the Hölder condition with exponent α for $|x| \leqslant \beta$), then the following theorem holds.

Theorem 1.1. If the solution of the system (1.2) in the class of functions $L_p(-1, 1)$, $1+\delta > p > 1$, $\delta > 0$ exists and is unique, then for any $\alpha \in (0, \pi)$ it has the form

$$q(\alpha x) = \omega^{(1)}(x)\,(1-x^2)^{-1/2},\quad q_1(k\alpha x+\pi) = \omega^{(2)}(x)\,(1-x^2)^{-1/2}$$
$$(\omega^{(i)}(x) \in C(-1, 1))$$

We shall not stop to prove the theorem since it is easily proved if the following property is used [5]

$$\int_{-1}^{1} \frac{\gamma(t)\sqrt{1-t^2}}{t-x}\,dt \in C_m(-1, 1), \qquad \text{if} \quad \gamma(t) \in H_m^{\alpha}(-1, 1)$$

Let us turn to the construction of an aysmptotic solution, for small α, of the system of integral equations (1.4). To do this, let us first represent $K(t)$ and $K(t-\pi)$ as

$$K(t) = \ln|t|\,F_1(t) + |t|\,F_2(t) + F_3(t) + a_{30} + a_{20}|t| - \ln|t|$$
$$K(t-\pi) = F_4(t) + a_{40},\qquad F_i(t) = \sum_{k=1}^{N} a_{ik}\,t^{2k}\qquad (t = \varphi - \theta) \tag{1.6}$$

and the right sides of (1.2) as

$$-\pi C\cos\theta + \frac{\pi\Delta}{R} f_1(\theta) = \pi\sum_{k=0}^{N} b_{1k}\theta^{2k},\qquad \pi C\cos\theta + \frac{\pi\Delta}{R} f_2(\theta+\pi) = \pi\sum_{k=0}^{N} b_{2k}\theta^{2k} \tag{1.7}$$

Let us seek the solution of the system (1.4) as [4]

$$\omega^{(i)}(x) = \sum_{l=0}^{N}\sum_{j=0}^{m}\omega_{lj}^{(i)}(x)\,\alpha^{l}\ln^{j}\alpha \tag{1.8}$$

Substituting (1.6) - (1.8) into (1.4), and equating terms in identical powers of α and $\ln\alpha$, we obtain equations for the successive determination of $\omega_{lj}^{(i)}(x)$, which yield when solved

$$\omega_{00}^{(i)}(x) = \pi^{-1}P_i,\ \omega_{10}^{(i)}(x) = [-b_{i2}(1-2x^2) + \pi^{-2}a_{20}S_1(x)\,P_i]\,\delta_i$$

$$\omega_{20}^{(i)}(x) = \pi^{-1}(1-2x^2)\,(P_jka_{41} + P_ia_{11}\delta_i\ln\delta_i) - 2\pi^{-1}a_{20}\,b_{i2}\delta_i^2\,S_4(x) + \pi^{-1}P_i \times$$
$$\times [(0.8069\,a_{11} + a_{31})(1-2x^2) + 32\pi^{-4}a_{20}^2\,(S_2(x) - 0.1508)]\,\delta_i^2$$

$$\omega_{21}^{(i)}(x) = \pi^{-1}P_ia_{11}(1-2x^2)\,\delta_i^2,\ \omega_{31}^{(i)}(x) = P_if_{31}(x)\,\delta_i^3 \tag{1.9}$$

$$\omega_{41}^{(i)}(x) = \pi^{-1}P_if_{41}(x)\delta_i^4,\quad \omega_{30}^{(i)}(x) = \{(f_{31}(x)\ln\delta_i + f_{30}(x))\,P_i + (^3/_2\,a_{11}b_{i2} + 4b_{i4}) \times$$
$$\times(x^4 - {}^1/_2\,x^2 - {}^1/_8) + [{}^1/_2a_{11}(1-2x^2) - {}^4/_3\pi^{-4}a_{20}^2\,N_3(x)]\,b_{i2}\}\,\delta_i^3 + 2\pi^{-3}a_{41}a_{20}P_jk\delta_i$$

$$\omega_{40}^{(i)} = \{P_i\,(f_{40}(x) + f_{41}(x)\ln\delta_i) + [{}^8/_3\,a_{20}^3\pi^{-6}N_5(x) - a_{20}a_{11}\pi^{-1}N_3(x) - a_{31}\pi^{-1}N_1(x)]\,b_{i2} -$$
$$- a_{20}\pi^{-1}\,[(1-2x^2)\,S_4(x) + {}^9/_{16}]\,b_{i4}\}\,\pi^{-1}\delta_i^4 + \pi^{-1}k\,P_j\{{}^4/_3\,\pi^{-4}\,a_{11}a_{20}N_3(x) +$$
$$+ (^3/_2a_{41}a_{11} + 4a_{42})(-x^4 + {}^1/_2\,x^2 + {}^1/_8) - {}^1/_2\,a_{11}a_{41}(1-2x^2)]\,\delta_i^2 + 6\pi^{-2}\alpha_{42}P_jk\delta_j^2$$

Here

$$f_{31}(x) \doteq 2\pi^{-3}a_{11}a_{20}S_4(x),\ f_{30}(x) = \pi^{-3}\left({}^{8}/_{9}\, a_{11}a_{20}S_3(x) + [6a_{21}(1+2x^2) - 19.3\pi^{-4} \times \right.$$
$$\left. \times a_{20}^2] S_1(x) + [9a_{21} + (1.614\, a_{11} + 2a_{31})\, a_{20}]\, S_4(x) + {}^{8}/_{3}\, a_{21} + 64\pi^{-4}a_{20}^3 S_5(x)\right)$$

$$f_{41}(x) = 4\,(x^4 + x^2 - {}^{7}/_{8})\, a_{12} + {}^{3}/_{2}\,(x^4 - 2x^2 + {}^{5}/_{8})\, a_{11}^2 - {}^{4}/_{3}\, \pi^{-4}a_{11}a_{20}^2 N_9(x)$$

$$f_{40}(x) = -4\,(x^4 + x^2 - {}^{7}/_{8})a_{32} - (5.561x^4 - 0.4384x^2 - 1.866)a_{12} + 8\pi^{-4}a_{20}a_{21}N_2(x) +$$
$$+ {}^{2}/_{3}\,\pi^{-4}\, a_{20}^2\,(1.614a_{11} + 2a_{31})\, N_9(x) + 64\pi^{-8}a_{20}^4 N_6(x) - (1.614a_{11} + 2a_{31})\,(x^4 - 2x^2 + {}^{5}/_{8}) +$$
$$+ 16\pi^{-4}a_{11}a_{20}^2 N_4(x)$$

$$N_1(x) = {}^{4}/_{15} + (x^2 - {}^{5}/_{2})\; S_4(x) - 3S_1(x),\ N_2(x) = -{}^{4}/_{3}\, S_1(x) + 12\,(1 + x^2) \times$$
$$\times S_2(x) + 6S_7(x) - {}^{3}/_{4}\, S_8(x) + 4.189x^2 - 5.570$$

$$N_3(x) = {}^{5}/_{9}\,(1 - 2x^2) - S_1(x) + {}^{2}/_{3}\,(x^2 - {}^{5}/_{3})\; S_4(x) + {}^{8}/_{45} \qquad (1.10)$$

$$N_4(x) = S_8(x) + (0.08312x^2 - 1.208)S_4(x) - 0.2494S_1(x) - 0.0264x^6 + 0.09928x^4 -$$
$$- 0.5635\, x^2 - 0.2986$$

$$N_5(x) = (0.8896 - 0.1657x^2 - 0.05714x^4)\, S_6(x) - (3.146 - 1.156x^2 + 0.01905x^4) \times$$
$$\times S_4(x) - 0.0192S_1(x) + 0.3723 + 0.01523x^2,\ N_6(x) = (0.4583 - 0.1658x^2 + 0.003168x^4) \times$$
$$\times S_4(x) - (0.1245 - 0.01435x^2 - 0.003502\, x^4)S_6(x) + 0.00478S_1(x) - 0.05016 -$$
$$- 0.002534x^2,\ \ N_7(x) = (0.2360 + 0.06605x^2)S_4(x) + 0.3948 - 1.308\, x^2 + 0.8207x^4 +$$
$$+ 0.08315\;(1 - x^2)^2 \ln^2 (1 - x)\; (1 + x)^{-1}$$

$$N_8(x) = 9S_4(x) + 6\,(1 + 2x^2)\, S_1(x) + {}^{8}/_{3},\ N_9(x) = 2S_4(x) - S_6(x)$$

$$U_{2n+2}(x) + U_{2n-1}(x) + 2\,(1 - 2x^2)\, U_{2n+1}(x) - 8x \,/\, (2n + 1) = 0$$

$$-U_{-1}(x) = U_1(x) = -\ln(1 + x)\,(1 - x)^{-1}$$

$$S_6(x) = {}^{3}/_{2} + (1 - 2x^2) + {}^{1}/_{2}\,(1 - x^2)^2\;(\ln^2\;(1 - x)(1 + x)^{-1} - \pi^2)$$

$$S_7(x) = (1 - x^2)\sum_{k=1}^{\infty}\left[\frac{3U_{2k}(x)}{(4k^2 - 9)^2} - \frac{4xU_{2k-1}(x)}{(2k + 1)^2\,(2k - 3)^2}\right]$$

$$S_8(x) = 96\,(1 - x^2)\sum_{k=1}^{\infty}\frac{U_{2k}(x)}{(4k^2 - 1)^2\,(4k^2 - 9)^2}$$

$$\delta_i = \begin{cases} 1, & \text{if} \quad i = 1 \\ k, & \text{if} \quad i = 2 \end{cases} \quad i \neq j;\ i,\ j = 1, 2$$

The expressions for the functions $S_i(x)$, $i = 1, 2, \ldots, 5$, as well as $U_{2k}(x)$ are presented in [4], see formula (1.15). Given there also are tables of the functions $S_i(x)$, $i = 1, 2, \ldots, 5$ for values of $|x| \leqslant 1$ with an $h = 0.1$ spacing.

Thus, an asymptotic solution of the system (1.4), or equivalently, the differentiated system (1.2), has been obtained. Evidently, the class of solutions of (1.2) will include the whole class of solutions of the original system (1.2), as well as solutions, extraneous to the system (1.2), which satisfy the solution of (1.2) with an arbitrary constant on the right side. Utilizing the arbitrariness latent in the solution of (1.4), it can be demanded that this solution should satisfy the original system (1.2). We hence obtain equations to determine the arbitrary constants P_i.

Let us now determine the P_i. To do this let us substitute the values found for $\omega^{(i)}(x)$ in the system (1.2). We then put $\theta = 0$, and by evaluating all the integrals we obtain a system of equations to determine the P_i, namely:

$$P_1 \sum_{l=0}^{4} (A_{l0} + A_{l1} \ln\alpha)\, \alpha^l + P_2 \left[\sum_{l=0}^{4} B_{l2} \alpha^l - {}^1/_4\, a_{41}{}^2 k^2 \alpha^4 \ln k\alpha \right] = \sum_{l=-1}^{3} C_{l1} \alpha^l - {}^1/_4\, a_{11} b_{12} \pi \alpha^3 \ln \alpha \tag{1.11}$$

$$P_1 \left[\sum_{l=0}^{4} B_{l1} \alpha^l - {}^1/_4\, a_{41}{}^2 k \alpha^4 \ln \alpha \right] + P_2 \sum_{l=0}^{4} (A_{l0} + A_{l1} \ln k\alpha)\, (k\alpha)^l =$$

$$= \sum_{l=-1}^{3} C_{l2} \alpha^l - {}^1/_4\, \pi a_{11} b_{22} k^3 \alpha^3 \ln ka$$

Here

$$A_{00} = a_{30} + \ln 2, \quad A_{10} = 0.8106 a_{20}, \quad A_{20} = a_{31} + 0.3069 a_{11} - 0.03288 a_{20}{}^2$$

$$A_{30} = 1.442 a_{21} - 0.1454 a_{11} a_{20} - 0.1802 a_{31} a_{20} - 0.01775 a_{20}{}^3$$

$$A_{40} = 2.025 a_{32} + 1.066 a_{12} - (0.8545 a_{11} + 0.6254 a_{31} - 0.0220 a_{20}{}^2)\, 10^{-1} a_{20}{}^2 - 0.3004 a_{20} a_{21} -$$

$$-0.1628\, a_{11}{}^2 - 0.4103 a_{11} a_{31} - 0.25\, (a_{31}{}^2 + k^2 a_{41}{}^2)$$

$$A_{01} = -1, \ A_{11} = 0, \ A_{21} = a_{11}, \ A_{31} = -0.1801 a_{11} a_{20}, \ A_{41} = 2.025 a_{12} - 0.4039\, a_{11}{}^2 -$$

$$- 0.0653 a_{11} a_{20}{}^2 - 0.5 a_{31} a_{11}, \ B_{0i} = k^{-1}\, a_{40} \delta_i{}^2 \tag{1.12}$$

$$B_{1i} = 0, \ B_{2i} = {}^1/_2 a_{41} (1 + k^2)\, k \delta_j{}^{-2}, \ B_{3i} = -\, 0.09006 a_{41}\, a_{20}\, k(1+k^2) \delta_j{}^{-2}, \ C_{0i} = 0$$

$$B_{4i} = -\, a_{41}\, (0.2017\, a_{11} + 0.25\, a_{31} + 0.04927 a_{20}{}^2),\ \delta^{-2}{}_j\ \ k\, (1 + k)^2, \ C_{-1i} = \pi b_{i0} \delta_i{}^{-1}$$

$$C_{1i} = -\, {}^1/_2\, \pi b_{i2} \delta_i, \ C_{2i} = -0.2829 a_{20} b_{i2}\, \delta_i{}^2, \ C_{3i} = -[(0.6337 a_{11} + 0.06051 a_{20}{}^2 + 0.7854 a_{31}) \times$$

$$\times\, b_{i2} - 1.178\, b_{i4}]\ \ \delta_i{}^3 - {}^1/_4\, \pi\, a_{41}\, k b_{j2} \delta_j \ (i \neq j; \ i, \ j = 1{,}2; \ \delta_1 = 1, \ \delta_2 = k)$$

And finally, we obtain the following formula to determine the Q_j :

$$Q_i = R\alpha\delta_i \{P_i\ [1 - {}^1/_4\, \alpha^2 \delta_i{}^2 + {}^4/_9\, \pi^{-2}\, a_{20}\, \alpha^{-2} \delta_i{}^3 - (0.1008 a_{11} + 0.125 a_{31} + 0.02484 a_{20}{}^2 -$$

$$- {}^1/_{64} - {}^1/_8\, a_{11} \ln \delta_i)\, \alpha^4 \delta_i{}^4 - {}^1/_8\, k a_{41} P_j\, \alpha^4 \delta_i{}^4 - {}^1/_8\, \pi\ b_{i2} \alpha^3 \delta_i{}^3\} \ (i \neq j, \ i, \ j = 1{,}2) \tag{1.13}$$

Formula (1.13) yields the value of the force referred to unit length of the disk. In order to obtain the force which must be applied to a disk of length l, the values given by (1.13) must be multiplied by l.

Let us write down a number of the coefficients a_{ik} in (1.6)

$$a_{11} = 0.5, \ a_{12} = -\, {}^1/_{24}, \ a_{20} = 0.25 \pi a, \ a_{21} = -\, {}^1/_{24} \pi a, \ a_{30} = -0.5 - b$$

$$a_{31} = 0.5({}^1/_{12} + b - 0.5a), \ a_{32} = {}^1/_{24}\, (a - b - {}^{113}/_{240}) \tag{1.14}$$

$$a_{40} = -\, 0.5 + \ln 2 + b, \ a_{41} = 0.5\, (0.5a - b - \ln 2 - 0.25), \ a_{42} = {}^1/_{24}\, (b - a + \ln 2 +$$

$$+ {}^{19}/_{16}), \ (a = (1 - 2\nu)\, (1 - \nu)^{-1}, \ b = {}^1/_{16}\, (5 - 8\nu)\, (1 - \nu)^{-2})$$

The general scheme of a computation by means of the formulas proposed is the following:

1. We find a_{ik} by means of (1.14) by assigning the ν .

2. We find values of $P_i\ (i = 1{,}2)$ from the system (1.11), by giving α and k and utilizing (1.12), and then we find $Q_i\ (i = 1{,}2)$ from (1.13).

3. We find the arbitrary constant C from the condition $Q = Q_2 - Q_1$ by giving Q . Substituting C into (1.13), we find Q_2, and also $Q_1 = Q_2 - Q$.

4. Substituting C into the found values of P_i and (1.9), we find the stresses by means of (1.8), (1.5).

The computation is thereby terminated if we seek a solution of the system not bounded at the edges of the stamps. If the solution bounded at the edges of one, or both stamps, is to be found, then by requiring $\omega^{(i)}(1) = 0$, $i = 1$ (or $i = 2$, depending on which of the stamps the solution, bounded at the edges, is desired for), or $i = 1,2$, we obtain, respectively, one or two equations imposing definite conditions on the function $f_i(\theta)$, $i = 1$ (or 2), or on the function $f_i(\theta)$, $i = 1, 2$. As a rule, in this case one equation, or a system of equations, is obtained to determine the settling under the stamp at the point $\theta = 0$ or $\theta = \pi$, or at both these points.

Remark 1.1. The domain (domains) of contact cannot be given for the determination of the bounded solutions, but the forces Q_1, Q_2 can be given. In this case, the domain (domains) of contact can be determined from the condition of boundedness of the solution at the edges of the stamp (stamps). However, a complex transcendental equation (system of transcendental equations) in the angle (angles) of contact must be solved; hence, the proposed computational scheme assumes assignment of the angle (angles) of contact. The angle (angles) of contact for a given force Q_1, or Q_2, can always be determined by constructing a dependence of the force Q_1, or Q_2, on the angle (angles) of contact and on the force Q.

Remark 1.2. If there is one stamp, or two stamps of identical shape, it is then necessary to consider just one equation of the system (1.2). In the first case it is hence necessary to put $P_2 = k = 0$, and in the second case $P_1 = P_2$, $Q = C = 0$.

Remark 1.3. Let us consider a system of integral equations of rather more general form than the system (1.2) considered herein, namely:

$$\int_{-\alpha}^{\alpha} (-\ln|x-t| + {}^1/_2\mu_1|x-t| + K_1(x-t))\, q(t)\, dt + \\ + \int_{-k\alpha}^{k\alpha} q_1(t)\,[K_{14}(x-t) + b_1]\, dt = f_1(x), \quad |x| < \alpha \tag{1.15}$$

$$\int_{-k\alpha}^{k\alpha} (-\ln|x-t| + {}^1/_2\mu_2|x-t| + K_2(x-t))\, q_1(t)\, dt + \\ + \int_{-\alpha}^{\alpha} q(t)\,[K_{24}(x-t) + b_2]\, dt = f_2(x), \quad |x| < k\alpha$$

Here

$$K_i(\tau) = \ln|\tau|\, F_{i1}(\tau) + |\tau|\, F_{i2}(\tau) + F_{i3}(\tau) + c_i$$

$$F_{il}(\tau) = \sum_{m=1}^{N} a_{ilm}\tau^m, \quad \tau = x - t, \quad a_{111} = a_{211} = 0 \tag{1.16}$$

It is easy to note that the method utilized herein to solve the system (1.2) is also applicable to the system (1.15), as well as a system obtained from (1.15) by differentiation once with respect to x, , i.e. to a system of integral equations of the form

$$\mu_1 \int_{-\alpha}^{x} q(t)\, dt + \int_{-\alpha}^{\alpha} \frac{q(t)}{t-x}\, dt = f'_1(x) - \int_{-\alpha}^{\alpha} q(t)\, K_1'(x-t)\, dt - \\ - \int_{-k\alpha}^{k\alpha} q_1(t)\, K_{14}'(x-t)\, dt + \frac{\mu_1}{2} \int_{-\alpha}^{\alpha} q(t)\, dt, \quad |x| < \alpha \tag{1.17}$$

$$\mu_2 \int_{-k\alpha}^{x} q_1(t)\,dt + \int_{-k\alpha}^{k\alpha} \frac{q_1(t)}{t-x}\,dt = f_2(x) - \int_{-k\alpha}^{k\alpha} q_1(t) K_2'(x-t)\,dt -$$
$$- \int_{-\alpha}^{\alpha} q(t) K_{24}'(x-t)\,dt + \frac{\mu_2}{2} \int_{-k\alpha}^{k\alpha} q(t)\,dt, \quad |x| \leqslant k\alpha$$

This latter follows from the fact that, as has already been remarked earlier for the system (1.2), it is necessary to solve a system of the form (1.17) first before finding the solution of the system (1.15).

Let us note that if $F_{11}(\tau) = 0$ in (1.16), then the solution of the system (1.15), or (1.17), must be sought as a power series in just α.

Remark 1.4. Let us consider an integro-differential equation of the form

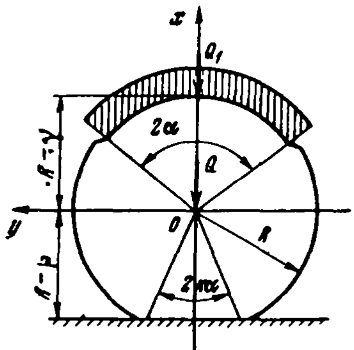

Fig. 2

$$\mu\gamma(x) + \int_{-\alpha}^{\alpha} \frac{\gamma'(t)}{t-x}\,dt = f(x), \quad |x| \leqslant \alpha \qquad (1.18)$$

If we put $\gamma'(x) = q(x)$ in (1.18), then $\gamma(x) = \int_{-\alpha}^{x} q(t)\,dt + C$, and (1.18) can be written thus

$$\mu \int_{-\alpha}^{x} q(t)\,dt + \int_{-\alpha}^{\alpha} \frac{q(t)}{t-x}\,dt = f(x) - \mu C$$

In form (1.19) agrees with the first of the equations of the system (1.17), and therefore, if α is small, the method elucidated herein can be applied to (1.19), and we hence obtain a solution of (1.18) also. The arbitrary constant C can be determined from the condition obtained if it is required that the solution of (1.19) satisfy the original equation (1.18).

2. Examples. Let

$$(1)\ f_1(\theta) = \gamma = \text{const}, \ f_2(\theta) = R + (R-\beta)\cos^{-1}\theta$$

In this case

$$b_{10} = \Delta\gamma R^{-1} - C,\ b_{12} = {}^1/_2\, C,\ b_{14} = -{}^1/_{24}\, C,\ b_{20} = \beta\Delta R^{-1} + C$$
$$b_{22} = -{}^1/_2\,(\Delta + \beta\Delta R^{-1} + C),\ b_{24} = {}^1/_{24}\,(C - 5\Delta - 5\,\beta\Delta\, R^{-1})$$

Let us put $\nu = 0.3$, $\alpha = 0.5$, $k = 0.1$; using the proposed scheme of computations we obtain

$$Q_2 = (3.728\Delta R + 5.656\Delta\gamma + 3.591\,Q)\,10^{-3}$$
$$\Delta\beta = 1.440\,Q_2 - 0.9593\,\Delta\gamma - 0.4023\,Q - 0.5873\ 10^{-3}\ \Delta R$$
$$CR = 0.2235\,Q - 0.3457\,Q_2 + 01.9544\,\Delta\gamma - 0.3667\cdot 10^{-4}\ \Delta R$$
$$(2)\ f_1(\theta) = \gamma_1,\ f_2(\theta) = \gamma_2,\ \gamma_i - \text{const}$$

In this case

$$b_{10} = \Delta\gamma_1 R^{-1} - C,\ b_{12} = {}^1/_2\, C,\ b_{14} = -{}^1/_{24}\, C$$
$$b_{20} = \Delta\gamma_2 R^{-1} + C,\ b_{22} = -{}^1/_2\, C,\ b_{24} = {}^1/_{24}\, C$$

We put $\gamma = 0.3,\ \alpha = 0.5,\ k = 1$, and we obtain

$$Q_2 = 0.428\,\pi\Delta\,(\gamma_1 + \gamma_2) + {}^1/_2\, Q,\quad CR = 0.15\ \pi\Delta\ (\gamma_1 - \gamma_2) + 0.04371\ Q$$

Now if $\gamma_1 = \gamma_2 = \gamma$, then $P_1 = P_2,\ Q = C = 0$; we then obtain

$$Q_1 = 0.856\,D,\ P_1 = 1.819\ (1.820)\,D,\ \omega^{(1)}\ (0) = 1.755\ (1.787)\,D$$
$$\omega^{(1)}\,(0.5) = 1.796\ (1.819)\,D,\ \omega^{(1)}\,(1) = 1.852\ (1.821)\,D,\ D = \pi\Delta\gamma$$

Values of these quantities calculated by the method expounded in [6] are presented in parentheses.

Table 1

x	N_1	N_2	N_3	$N_4\cdot 10^4$	N_5	$N_6\cdot 10^4$	N_7	N_8	N_9
0.0	−5.562	4.208	−1.987	−4324	−5.855	8472	1.024	19.66	4.934
0.1	−5.406	4.077	−1.916	−4179	−5.622	8140	0.9907	19.31	4.756
0.2	−4.949	3.698	−1.853	−3724	−4.948	7177	0.8920	18.26	4.235
0.3	−4.214	3.077	−1.377	−3018	−3.893	5647	0.7333	16.50	3.410
0.4	−3.239	2.165	−0.9546	−2097	−2.529	3709	0.5235	13.98	2.310
0.5	−2.076	1.199	−0.4746	−1052	−1.063	1594	0.2777	10.64	1.111
0.6	−0.789	0.687	−0.0172	−0019	0.4130	−0632	0.0067	6.429	−0.157
0.7	0.557	−0.751	−0.4707	1016	1.697	−2432	−0.2576	1.239	−1.327
0.8	1.886	−1.380	0.8277	1809	2.590	−3756	−0.4808	− 5.017	−2.219
0.9	3.133	−1.367	1.029	2737	2.887	−4217	−0.6108	−12.49	−2.606
1.0	4.267	−0.057	0.9553	2102	2.191	−3216	−0.4961	−21.33	−2.000

Values of the function $N_i\,(x),\ i = 1,2,\ldots,9$, determined by the relations (1.10), are presented in Table 1 for convenience in practical utilization of the results obtained herein.

In conclusion, let us note that, as computations have shown, the formulas obtained herein can successfully be used for values $\alpha \leqslant 0.6$.

BIBLIOGRAPHY

1. Aleksandrov, V.M. and Belokon', A.V., Contact problems of elasticity theory for cylindrical bodies. Annot. Repts. of the Third All-Union Congress on Theoretical and Applied Mech. "Nauka", 1968.
2. Muskhelishvili, N.I., Some Fundamental Problems of the Mathematical Theory of Elasticity. 5th Ed. Moscow, "Nauka", 1966.
3. Papkovich, P.F., Theory of Elasticity. Moscow-Leningrad, Oborongiz, 1939.
4. Aleksandrov, V.M. and Belokon', A.V., Asymptotic solution of a class of integral equations and its application to contact problems for cylindrical elastic bodies. PMM Vol.31, No.4, 1967.
5. Aleksandrov, V.M., On the approximate solution of a certain type of integral equations. PMM Vol.26, No.5, 1962.
6. Aleksandrov, V.M., Axisymmetric contact problem for an elastic infinite cylinder. Izv. Akad. Nauk SSSR, OTN, Mekh. i Mashinostr. No.5, 1962.

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